Math 255A' Lecture 26 Notes

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1 The Spectrum and The Spectral Radius

1.1 The spectrum of an element

Let $\mathscr{A}$ be a Banach algebra with identity 1. Recall that if $\|a-1\| < 1$, then a^{-1} exists and equals $\sum_{k\geq 0}(1-a)^k$. The **spectrum** of a is $\sigma(a) = \{z \in \mathbb{F} : z - a \text{ is not invertible in } \mathcal{A}\}$ (and similarly for right/left spectrum σ_r, σ_ℓ). The **resolvent** is $\rho(a) = \mathbb{F} \setminus \sigma(a)$.

Example 1.1. Let X be a compact, Hausdorff space. If $f \in C(X)$, then $\sigma(f) = f[X]$.

Example 1.2. Let X be a Banach space, and let $A \in \mathcal{B}(X)$. Then

$$\sigma(A) = \{\lambda \in \mathbb{F} : A - \lambda \textbf{ is not a bijection } X \to X\},$$

$$\sigma(A) = \{\lambda \in \mathbb{F} : \inf\{\|(A-\lambda)x : \|x\| = 1\} = 0\}.$$

Example 1.3. If $\mathbb{F} = \mathbb{R}$, we can have elements with empty spectrum. For example, take

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in M_2(\mathbb{R}).$$

If we take this as an element in $M_2(\mathbb{C})$, the spectrum is nonempty. So the spectrum depends on the space the element is sitting in.

Theorem 1.1. *If $\mathbb{F} = \mathbb{C}$ and $a \in \mathscr{A}$, then $\sigma(a)$ is a nonempty, compact subset of $\{z \in \mathbb{C} : |z| \leq \|a\|\}$.*

Proof. Consider $z - a = z(1 - a/z)$. If $|z| > \|a\|$, then $\|a/z\| < 1$. Then $(z-a)^{-1} = \frac{1}{z}(1 - \frac{a}{z})^{-1}$ exists. This tells us that $\sigma(a) \subseteq \{z \in \mathbb{C} : |z| \leq \|a\|\}$.

Also $\rho(a) = g^{-1}(\{\text{invertible elements}\})$, where $g(z) = z - a$ is continuous. Since the invertible elements form an open set, we have $\rho(a)$ is open. So $\sigma(a)$ is closed and bounded.

Consider the **resolvent function** $f : \rho(a) \to \mathscr{A}$ by $z \mapsto (z-a)^{-1}$. This is a continuous map from $\rho(a) \to \{\text{invertible elements in } \mathscr{A}\}$. If $|z| > \|a\|$, then

$$f(z) = \frac{1}{z}\left(1 - \frac{a}{z}\right)^{-1} = \frac{1}{z}\sum_{k\geq 0}\frac{a^k}{z^k},$$

so we can get

$$\|f(z)\| \leq \frac{1}{|z|}\sum_{k\geq 0}\frac{\|a\|^k}{|z|^k} = O(1/|z|) \qquad \text{as } |z|\to\infty.$$

If $z \in \rho(a)$, then

$$\frac{f(z+h)+f(z)}{h} \xrightarrow{h\to 0} ((z-a)^{-1})^2$$

(proven below). So we can say "f is holomorphic on $\rho(a)$."[1]

This shows that if $\sigma(a) = \varnothing$, then f is a holomorphic and bounded function. By a version of Liouville's theorem (proven below), f is constant. So $f = 0$. But this is a contradiction. $\square$

Lemma 1.1. *If $z \in \rho(a)$, then*

$$\frac{f(z+h)+f(z)}{h} \xrightarrow{h\to 0} ((z-a)^{-1})^2.$$

Proof. If $, y \in \mathscr{A}$ are invertible, then

$$\begin{aligned} x^{-1} - y^{-1} &= x^{-1}yy{-}1 - x^{-1}xy^{-1} \\ &= xx^{-1}(y-x)y^{-1}. \end{aligned}$$

This is called the **resolvent identity**. So

$$\begin{aligned} \frac{1}{h}[(z+h-a)^{-1} - (z-a)^{-1}] &= (z+h-a)^{-1}(z-a)^{-1} \\ &\xrightarrow{h\to 0} ((z-a)^{-1})^2. \end{aligned}$$

$\square$

Lemma 1.2 (Liouville's theorem for Banach-valued holomorphic functions)**.** *Let X be a Banach space. If $f : \mathbb{C} \to X$ is holomorphic and bounded, it is constant.*

Proof. For any $\varphi \in X^*$, $\varphi \circ f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded, so it is constant. $\square$

This trick is a common way to transfer results from complex-valued holomorphic functions to Banach-valued ones.

1.2 Spectral radius

Definition 1.1. Let $a \in \mathscr{A}$. The **spectral radius** of a is $r(a) := \sup\{|z| : z \in \sigma(a)\}$.

[1] This is a notion of holomorphic functions that take values in a Banach algebra.

Example 1.4. In $M_2(\mathbb{C})$, let

$$a = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then $a^2 = 0$, so $\sigma(a) = \{0\}$. So the spectrum of a is nonempty, but it has zero spectral radius.

Theorem 1.2 (spectral radius formula)**.** *Let $a \in \mathscr{A}$. Then*

$$r(a) = \lim_{n \to \infty} \|a^n\|^{1/n}.$$

Remark 1.1. Since the norm is submultiplicative, this is $\leq \|a^m\|^{1/m}$ for any m. So this equals $\inf_n \|a^n\|^{1/n}$.

Proof. ($\leq$): We know that $r(a) \leq \|a\|$. We claim that $\sigma(a^m) = \{z^m : z \in \sigma(a)\}$.[2] If $\lambda \in \mathbb{C}$, then $a^m - \lambda = \prod_{i=1}^m (a - \omega_i)$, where the ω_i are the m-th roots of λ. Since each $a - \omega_i$ is invertible, $a^m - \lambda$ is invertible. If $a^m - \lambda$ is invertible, then $(a - \omega_1)^{-1} = \prod_{i=2}^m (a - \omega_i)(a^m - \lambda)^{-1}$. This proves the claim and gives us $r(a^m) = r(a)^m$ for any m. So $r(a) = r(a^m)^{1/m} \leq \|a^m\|^{1/m}$ for all m.

($\geq$): Let $h(w) = (\frac{1}{w} - a)^{-1}$ for w such that $\frac{1}{w} \in \rho(a)$. Extend this so $h(0) = 0$. As before,

$$h(w) = w \sum_{k \geq 0} w^k a^k \qquad \forall |w| < \|a\|^{-1},$$

and h is holomorphic on $\{0\} \cup \{\frac{1}{z} : z \in \rho(a)\}$. Now we use a fact from complex analysis (which extends to this case): By Hadamard's formula for the radius of convergence of a series, the supremal R such that h has a holomorphic extension to the ball $B_{\mathbb{C}}(0, R)$ equals the radius of convergence of the series; this is $\lim_n 1/\|a^n\|^{1/n}$. So $\inf\{1/|z| : z \in \sigma(a)\} = \lim_n 1/\|a^n\|^{1/n}$. $\square$

Remark 1.2. Here is another way to show that the sequence converges. We have $\|a^{n+m}\| \leq \|a^n\| \cdot \|a^m\|$, so $\log \|a^{n+m}\| \leq \log \|a^n\| + \log \|a^m\|$. Now use Fekete's subadditive lemma.

Example 1.5. For $f \in L^2([0,1])$, the Volterra operator is

$$Vf(x) = \int_0^x f = \int_0^1 \mathbb{1}_{\{y \leq x\}} f(y)\, dy.$$

Then $\sigma(V) = \{0\}$, so $r(V) = 0$.

Proposition 1.1. *If $z \in \rho(a)$, then $\|(z-a)^{-1}\| \geq \frac{1}{\mathrm{dist}(z, \sigma(a))}$. In other words,* $\mathrm{dist}(z, \sigma(a)) \geq 1/\|(z-a)^{-1}\|$.

[2] This is a special case of the spectral mapping theorem, which we will discuss later.

If z is in the spectrum, $(z-a)^{-1}$ doesn't exist. This says that if z is close to the spectrum, then this blows up.

Proof. If $h \in \mathbb{C}$ with $|h| < \frac{1}{\|(z-a)^{-1}\|}$, then

$$z + h - a = (z-a)(h(z-a)^{-1} + 1)$$

is invertible. So $B(z, 1/\|(z-a)^{-1}\|) \subseteq \rho(a)$. $\square$

1.3 Riesz functional calculus

Here is a teaser for what we will discuss next time.

If $a \in \mathscr{A}$, then the resolvent map $f : \rho(a) \mapsto \mathscr{A}$ takes $z \mapsto (z-a)^{-1}$. Any holomorphic $f : G \to \mathscr{A}$ satisfies Cauchy's integral formula. As a result, if G is an open subset of $\mathbb{C}$ with $G \supseteq \sigma(a)$, then let $\Gamma = \gamma_1 \cup \cdots \cup \gamma_m$ wind once around any $z \in \sigma(a)$ and 0 times around any $z \in \mathbb{C} \setminus G$. Then if $f : G \to \mathbb{C}$ is holomorphic, define

$$f(a) = \frac{1}{2\pi i} \int_\Gamma f(z)(z-a)^{-1}\, dz \in \mathscr{A}.$$

This allows us to produce more elements of our Banach algebra.